Applied statistics: Coursework 1

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5th March 2019

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1 Task 1

1.1 Part (1)

In the given data were two out of 26 data points with an Al/Be ratio of more than 4.5. That means

$$\hat{p} = \frac{2}{26} = \frac{1}{13}$$

1.2 Part (2)

Using the following formula from the lecture we get the 95% confidence interval:

$$\hat{p} \pm 2 \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\frac{1}{13} \pm \underbrace{2 \cdot \sqrt{\frac{\frac{1}{13} \cdot \frac{12}{13}}{26}}}_{0.1045}$$

Our 95% confidence interval is [-0.0276,0.1814] which means that we are 95% sure that the true proportion lies between -0.0276 and 0.1814.

1.3 Part (3)

To get the 95% confidence interval via bootstrap I want to use the `bootci` function in MATLAB.

```
data = [3.75, 4.05, 3.81, 3.23, 3.13, 3.3, 3.21, 3.32, ...
4.09, 3.9, 5.06, 3.85, 3.88, 4.06, 4.56, 3.6, 3.27, ...
4.09, 3.38, 3.37, 2.73, 2.95, 2.25, 2.73, 2.55, 3.06];
parameter = @(y) length(find(y > 4.5))/length(y);

bootci(10000,{parameter, data},'alpha',0.05,'type',...
'percentile')
```

That gives the 95% confidence interval: [0,0.1923]

1.4 Part (4)

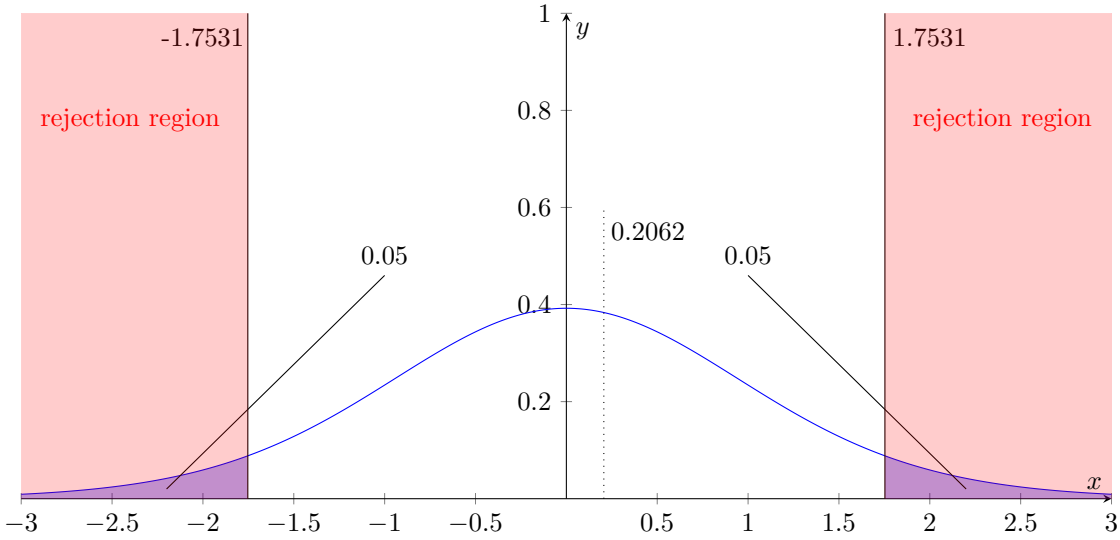
Yes, the confidence interval from the bootstrap procedure is more appropriate because it's not containing Al/Be ratios that are not possible like -0.0276. A negative ratio would suggest that there is a negative amount of data points in the sample which exceed 4.5. That is not possible.

2 Task 2

2.1 Part (1)

```
x = [-4.5, -1, -0.5, -0.15, 0, 0.01, 0.02, 0.05, ...
0.15, 0.2, 0.5, 0.5, 1, 2, 3];
m = mean(x);
s = std(x);
```

null hypothesis	H_0: $\mu = 0$
alternative hypothesis	H_A: $\mu \neq 0$
t-test for μ	$t = \frac{m-0}{\frac{s}{\sqrt{15}}} = \frac{0.0853}{\frac{1.6031}{\sqrt{15}}} = 0.2062$
rejection region	`tinv(0.05,15)` = -1.7531
conclusion	t is not in the rejection region so H_0 is accepted at the 10% significance level.



2.2 Part (2)

If we reduce the significance level our rejection region gets smaller. With $\alpha = 0.05$ the rejection region will start at `tinv(0.025,15)` = -2.1314. The t calculated in part (1) won't change $\Rightarrow$ our decision won't change too.

To get the type 2 error we use the MATLAB function `sampsizepwr` and $type\,2\,error = 1 - power$.

```
testtype = 't';
p0 = [0 1.6031];
p1 = 0.0853;
n = 15;
power = sampsizepwr(testtype,p0,p1,[],n)
```

This gives $power = 0.0542 \Rightarrow type\,2\,error = 0.9458$. This is the probability of wrongly accepting H_0 when it is false.

2.3 Part (3)

H_0: $\mu = 0$, normal distribution, small model M_S
H_A: $\mu \neq 0$, normal distribution, big model M_B
The log-likelihood function for normal distribution is

$$-\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j - \mu)^2 \tag{1}$$

Let's start with the MLEs for μ and σ in M_B:

$$\begin{aligned}\hat{\mu} &= \frac{1}{n}\sum_{j=1}^{n} x_j \\ &= 0.0853 \\ \widehat{\sigma^2} &= \frac{1}{n}\sum_{j=1}^{n}(x_j - \hat{\mu})^2 \\ &= 2.3986\end{aligned}$$

Maximum possible value for the log-likelihood $\xrightarrow{eq.\ (1)}$ -27.8457.
Now we'll calculate the MLE for σ in M_S:

$$\begin{aligned}\widehat{\sigma^2} &= \frac{1}{n}\sum_{j=1}^{n}(x_j - \hat{\mu})^2 \\ &= 2.3986\end{aligned}$$

Maximum possible value for the log-likelihood $\xrightarrow{eq.\ (1)}$ -27.8684.
Likelihood ratio test:

$$\begin{aligned}\chi^2 &= 2\Big(l(M_B) - l(M_S)\Big) \\ &= 0.0454\end{aligned}$$

It should be compared to χ^2(1 degree of freedom) since the difference in unknown parameters is equal to 1. The following piece of MATLAB code will calculate the p-value.

```
1  p = chi2cdf(0.0454,1,'upper')
```

The p-value is 0.8313 which means that we accept H_0: The small model M_S fits the data good enough. This is the same result as in part (1) and (2).

3 Task 3

3.1 Part (1)

First of all we need to prepare the data:

```
raw = load('input_data.txt');
data = reshape(raw,[1 500]); %produce a single vector
```

After that we do for every distribution (normal, exponential, uniform, lognormal, RAYLEIGH, gamma) the same procedure:

1. Estimate the parameter. This is often done with the function `<distribution>fit` but for estimating the parameters in the gamma distribution I used `fitdist(data', 'Gamma')` because `gamfit` doesn't work.
2. Creating the CDF with `makedist`.
3. Run the KOLMOGOROV-SMIRNOV test with `kstest`.

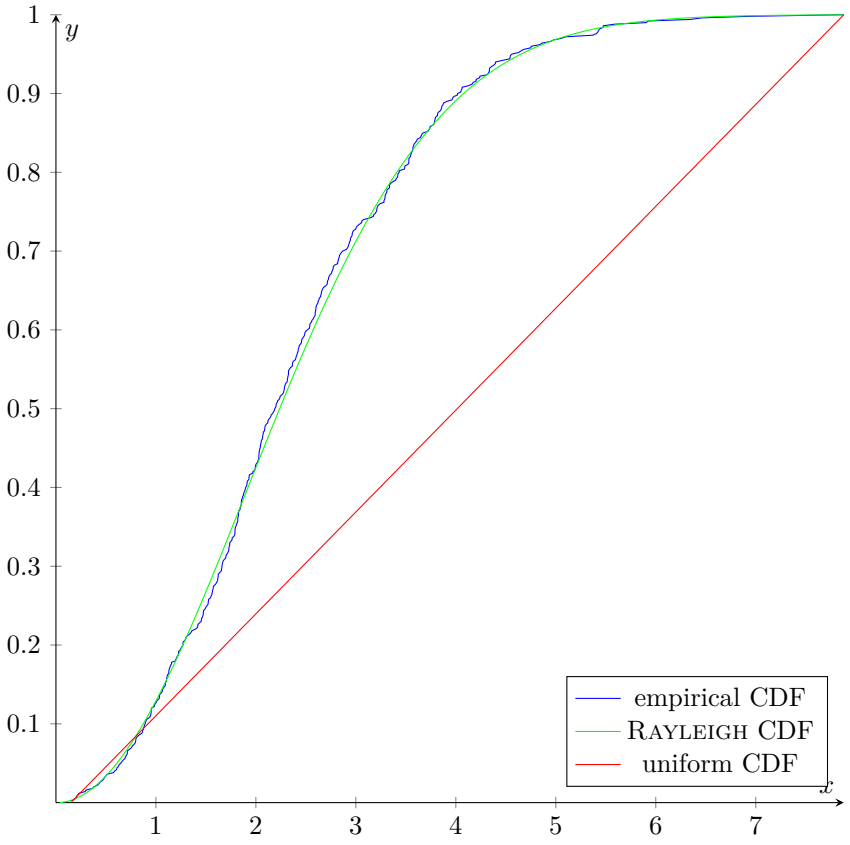
```
%normal distribution
[mu,sigma] = normfit(data)
norm_cdf = makedist('Normal','mu',mu,'sigma',sigma);
[h,p] = kstest(data,'CDF',norm_cdf)

%exponential distribution
mu = expfit(data)
exp_cdf = makedist('Exponential','mu',mu);
[h,p] = kstest(data,'CDF',exp_cdf)

%uniform distribution
[low,up] = unifit(data)
uni_cdf = makedist('Uniform','lower',low,'upper',up);
[h,p] = kstest(data,'CDF',uni_cdf)

%lognormal distribution
logmu = mean(log(data))
logsigma = std(log(data))
logn_cdf = makedist('Lognormal','mu',logmu,'sigma',logsigma);
[h,p] = kstest(data,'CDF',logn_cdf)

%rayleigh distribution
b = raylfit(data)
rayl_cdf = makedist('Rayleigh','b',b);
[h,p] = kstest(data,'CDF',rayl_cdf)

%gamma distribution
```

```
28 distribution = fitdist(data','Gamma');
29 a = distribution.a
30 b = distribution.b
31 gamma_cdf = makedist('Gamma','a',a,'b',b);
32 [h,p] = kstest(data,'CDF',gamma_cdf)
```

Running this gives the following output. The best fitting distribution is marked green, the worst red.

distribution	**estimated parameters**	KOLMOGOROV-SMIRNOV **test**	
		h	p
normal	$\mu = 2.3804$, $\sigma = 1.2486$	$h = 1$	$p = 0.0158$
exponential	$\mu = 2.3804$	$h = 1$	$p = 2.2618 \cdot 10^{-23}$
uniform	$lower = 0.1478$, $upper = 7.8807$	$h = 1$	$p = 1.5096 \cdot 10^{-72}$
lognormal	$\log(\mu) = 0.7050$, $\log(\sigma) = 0.6243$	$h = 1$	$p = 0.0017$
RAYLEIGH	$b = 1.9003$	$h = 0$	$p = 0.8939$
gamma	$a = 3.2378$, $b = 0.7352$	$h = 0$	$p = 0.2771$

3.2 Part (2)



4 Task 4

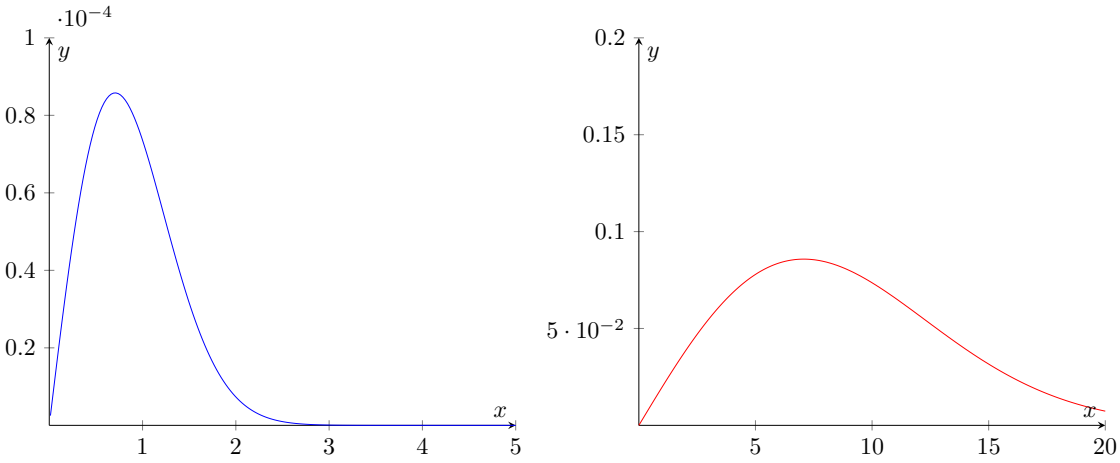
4.1 Part (1)

The probability density function $f(t)$ is

$$f(t) = \frac{2t \cdot \frac{\exp(-t^2)}{100}}{100} = \frac{t \cdot \exp(-t^2)}{5000}$$

After the PDF was changed we have

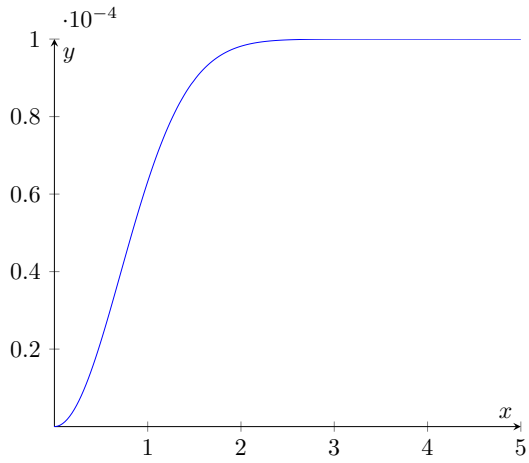
$$f(t) = \frac{2t \cdot \exp\left(\frac{-t^2}{100}\right)}{100}$$

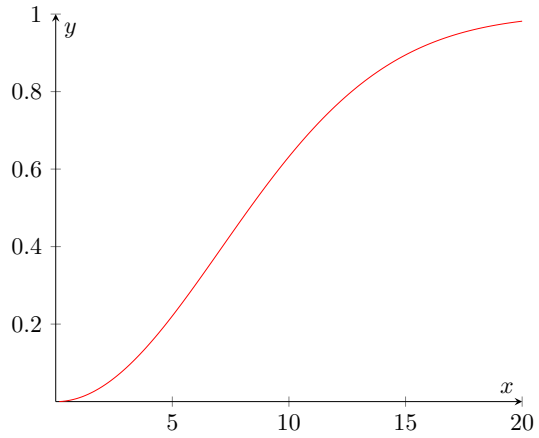


The cumulative distribution function $F(t)$ is then

$$\begin{aligned} F(t) &= \int_0^t f(\xi)\,\mathrm{d}\xi \\ &= \int_0^t \frac{\xi \cdot \exp(-\xi^2)}{5000}\,\mathrm{d}\xi \\ &= \frac{\exp(-t^2)\Big(\exp(t^2) - 1\Big)}{10000} \end{aligned}$$

$$\begin{aligned} F(t) &= \int_0^t f(\xi)\,\mathrm{d}\xi \\ &= \int_0^t \frac{2\xi \cdot \exp\left(\frac{-\xi^2}{100}\right)}{100}\,\mathrm{d}\xi \\ &= 1 - \exp\left(\frac{-t^2}{100}\right) \end{aligned}$$

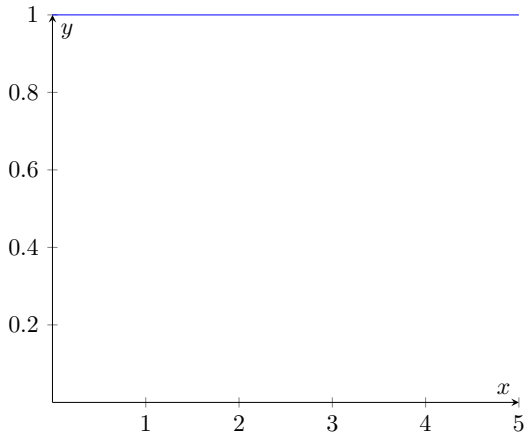


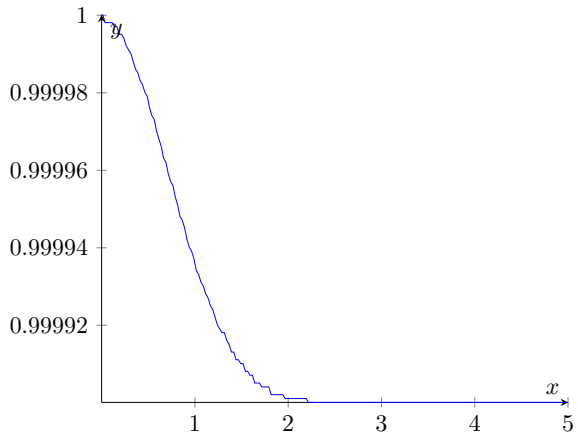


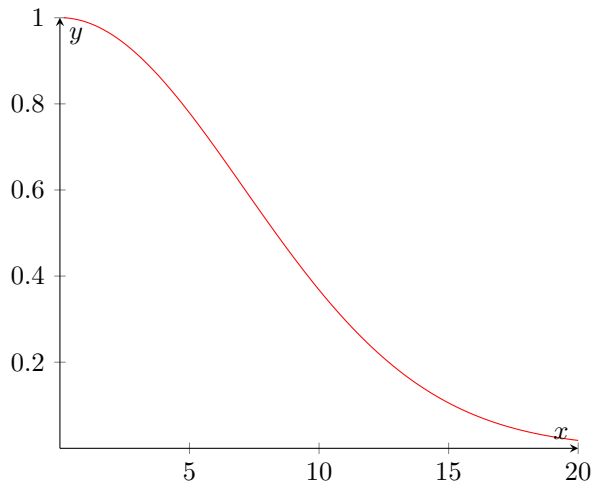
For the survival function we get

$$\begin{aligned} R(t) &= 1 - F(t) \\ &= \frac{\exp(-t^2) + 9999}{10000} \end{aligned}$$

$$\begin{aligned} R(t) &= 1 - F(t) \\ &= \exp\left(\frac{-t^2}{100}\right) \end{aligned}$$





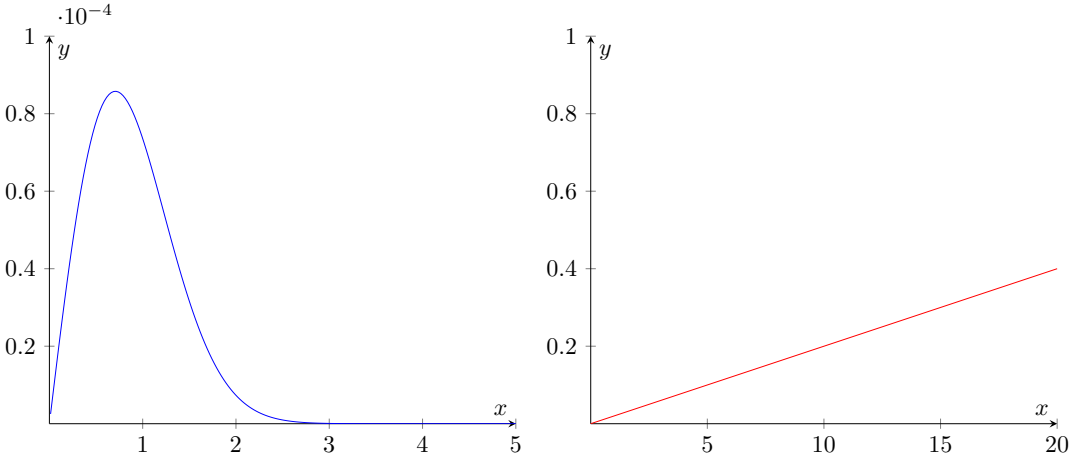


To get the reliability of the component at $t = 7$ we simply evaluate $R(7)$ which is 0.9999 (0.6126).

The hazard function is defined as

$$\begin{aligned} h(t) &= \frac{f(t)}{1 - F(t)} \\ &= \frac{2t}{9999 \cdot \exp(t^2) + 1} \end{aligned}$$

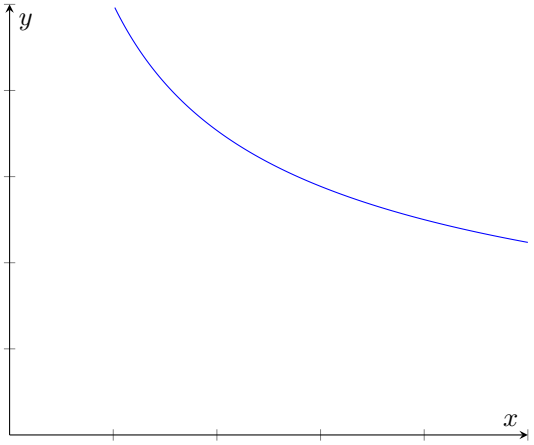
$$\begin{aligned} h(t) &= \frac{f(t)}{1 - F(t)} \\ &= \frac{t}{50} \end{aligned}$$



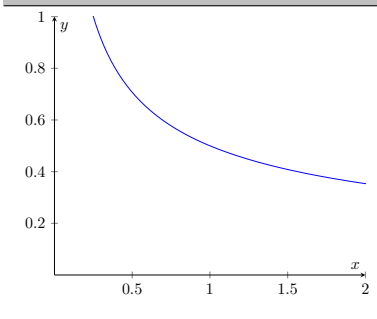
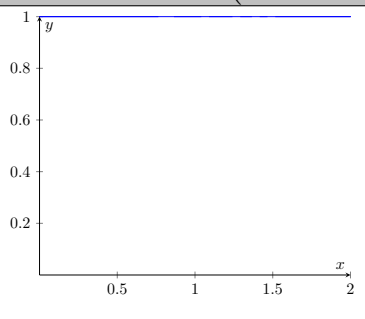
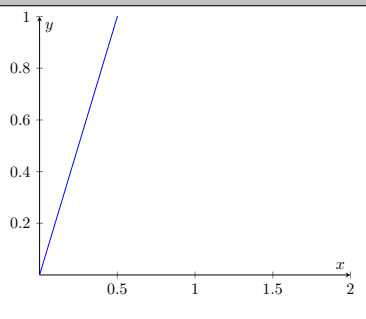
The hazard function describes how an item ages where t affects the risk of failure. It is the frequency with which the item fails, expressed in failures per unit of time.

4.2 Part (2)

Given $h(x) \sim (\sqrt{x})^{-1}$ we will try to find out the *shape*-parameter of the WEIBULL distribution first.



Comparing this graph to graphs of the hazard function with different *shape*-parameters we see that $shape = 0.5$ fits best.

$shape = 0.5$	$shape = 1$	$shape = 2$
Hazard function $\left(h = \frac{\text{PDF}}{1-\text{CDF}}\right)$		
		

To get the *scale*-parameter of the distribution we use the other provided information:

$$\begin{aligned} 5 &= \mu \\ &= scale \cdot \Gamma\left(1 + \frac{1}{shape}\right) \\ &= scale \cdot \Gamma(3) \\ \Rightarrow scale &= \frac{5}{2} \end{aligned}$$

Let's build the survival function:

$$\begin{aligned} R(t) &= 1 - \left(1 - \exp\left(-\sqrt{\frac{x}{5/2}}\right)\right) \\ &= \exp(-\sqrt{x} \cdot \sqrt{2.5}) \end{aligned}$$

That mean that the probability of surviving 6 years (30 years) is $R(6) = 0.0208$ ($R(30) = 0.0002$).