tangent space of these matrix Lie groups with the vector space needed for the Lie algebra. Continuing our program, we next introduce a basis for the tangent space.

Let $\mathfrak{B} = \{\omega_a\},\ a = 1, \ldots, D$ be the basis for $\mathcal{T}_e(\mathcal{G})$. We obtain D independent nowhere–vanishing vector fields on $\mathcal{G}$ such that

$$\Omega_a(g) = L_g^*(\omega_a). \tag{4.67}$$

This is already a very strong constraint on $\mathcal{M}(G)$ because it says that there exists no other vector field on $\mathcal{M}(G)$ that may be constructed independently of the basis vector, i.e. all vectors on $\mathcal{M}(G)$ have a basis decomposition.

To see how this manifests itself on any given manifold consider the following examples.

Example. The so–called **hairy ball theorem** says that any smooth vector field on $\mathcal{S}^2$ has at least two zeros. Thus $\mathcal{M}(G) \neq \mathcal{S}^2$. In fact for $\dim(G) = 2$, assuming G is compact, and $G \simeq U(1) \times U(1)$, the group manifold is actually a torus, $\mathcal{T}^2$. This is the only Lie group of dimension 2.

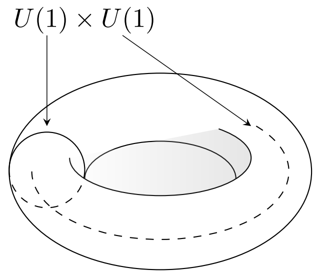

Figure 3. Group manifold of $U(1) \times U(1)$, the Torus, $\mathcal{T}^2$.

Definition 4.16. $\Omega_a(g)$ with $a = 1, \ldots, D$ are called **left–invariant vector fields** on $\mathcal{G}$, if they obey

$$L_h^*[\Omega_a(g)] = L_h^* \circ L_g^*(\omega_a) = L_{hg}^*(\omega_a) = \Omega_a(hg) \tag{4.68}$$

4.3.1 Lie Algebras for Matrix Lie Groups

For matrix Lie groups, $\mathcal{G} \subset \mathrm{Mat}_n(\mathbb{F})$ where $n \in \mathbb{N}$, and the fields are either real or complex. Then $\forall h \in \mathcal{G}, \forall X \in \mathcal{L}(\mathcal{G})$, we have

$$L_h^*(X) = hX \in \mathcal{T}_h(\mathcal{G}) \tag{4.69}$$

where the product is matrix multiplication.

Proof. The proof is fairly straightforward. To show that $hX \in \mathcal{T}_h(G)$, we need to check if hX is in the vector space of the Lie algebra. Starting with a curve and some initial conditions

$$C : t \in \mathbb{R} \mapsto g(t) \in \mathcal{G},\ g(0) = e,\ \dot{g}(t)\Big|_{t=0} = X \tag{4.70}$$

one finds near $t = 0$, there exists a Taylor series expansion

$$g(t) = \mathbb{I}_n + tX + \mathcal{O}(t^2). \tag{4.71}$$

Let C' be a new curve such that

$$C' : t \in \mathbb{R} \mapsto h(t) = hg(t) \in \mathcal{G} \tag{4.72}$$

which is guaranteed by closure of the group. Again neat $t = 0$, we find the expansion of $h(t)$ to be

$$h(t) = h\mathbb{I}_n + thX + \mathcal{O}(t^2). \tag{4.73}$$