Definition 4.13 (Injective). Left (and right) translations are **injective**,

$$\{\forall g, g' \in \mathcal{G} | L_h(g) = L_h(g') \iff g = g'\} \tag{4.62}$$

Lemma 4.4. L_h and R_h are diffeomorphisms of $\mathcal{M}(\mathcal{G})$.[15]

[15] *i.e. there exists a smooth bijection with a smooth inverse.*

Consider a manifold that contains the identity element, e. On this manifold, let the coordinates $\{\theta^i\}$, $i = 1, \ldots, D$ in some region contain the identity and a dependence of the group's elements on the parameter, θ, such that $g \equiv g(\theta) \in \mathcal{M}(\mathcal{G})$, and $g(0) = e$. Let $g' = g(\theta') = L_h(g) = hg(\theta)$. In these coordinates, L_h is specified by D real functions such that $\theta'^i = \theta'^i(\theta)$ with $i = 1, \ldots, D$.

Remark. L_n being a diffeomorphism implies that the Jacobian matrix,

$$J^i_j(\theta) = \frac{\partial \theta'^i}{\partial \theta^j} \tag{4.63}$$

exists and is invertible, i.e. $\det(J) \neq 0$.

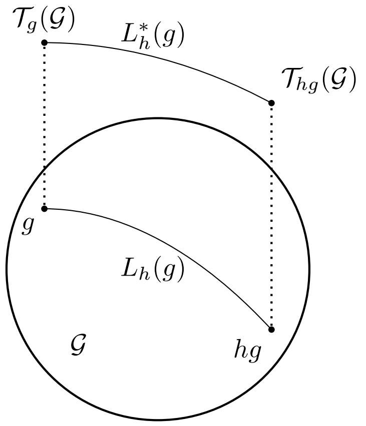

Figure 2. Visual of a push forward.

Diffeomorphism are known to generate a curve on a manifold connecting the points involved in a mapping. At each point on the curve, one can then assign a tangent vector and, more generally, a tangent space. These tangent space are also connected via a map known as the push–forward that are, in fact, induced by the existence of the left and right translation diffeomorphisms.

Definition 4.14. A left **push–forward**, $L^*_h : \mathcal{T}_g(G) \to \mathcal{T}_{hg}(G)$ is the map

$$V = v^i \frac{\partial}{\partial \theta^i} \in \mathcal{T}_g(G) \mapsto V' = v'^i \frac{\partial}{\partial \theta'^i} \in \mathcal{T}_{hg}(G) \tag{4.64}$$

with $v'^i = J^i_j(\theta) v^j$. Similarity for R^*_h.

From the push–forward definition, we can define a vector field generally and connect every point in the tangent space as a push–forward of the identity element.

Definition 4.15. A **vector field** is a map $\mathcal{V}$ which maps every $g \in \mathcal{G}$ to a vector $V_g \in \mathcal{T}_g(\mathcal{G})$ at g. In coordinates,

$$V(\theta) = V^i(\theta) \frac{\partial}{\partial \theta^i} \in \mathcal{T}_{g(\theta)}(\mathcal{G}) \tag{4.65}$$

is a vector field that is smooth if the components $v^i(\theta) \in \mathbb{R}$, $i = 1, \ldots, D$ are differentiable.

With regards to left and right translations, starting from a tangent vector at the identity, $\omega \in \mathcal{T}_e(\mathcal{G})$, we may define a vector field

$$\Omega(g) = L^*_g(\omega) \tag{4.66}$$

for all $g \in \mathcal{G}$. As L^*_g is smooth and invertible, $\Omega(g)$ is smooth and since the Jacobian is invertible, $\Omega(g)$ is also non-vanishing. This generalizes our result from earlier. We may now describe every point on the manifold as a push–forward of the identity.

Naturally, we want to tie in how this aids in developing a Lie algebra from Lie group. As we now have a vector space, we are just a short skip and step away from identifying the