1 Group Theory

Why groups? What is a group? How are groups helpful in discussing transformations and symmetries?

Often when we want to study transformations, we run into the problem of classifying the set of possible transformations. This is where groups come into play. Simply put, a group consists of a set of elements with a multiplication law which follows certain axioms. Groups are a compact way of generalizing the set of allowed transformations. This way, by simply describing the group, one can encompass the transformations and their corresponding symmetries. In fact, *every symmetry has an associated group encompassing it.* To make this discussion concrete, let's consider an example first and then delve into the mathematics.

Example (Rotation Transformations).
Let $\mathbf{x} \in \mathbb{R}^2$ be a two dimensional column vector with real entries. A 2×2 rotation matrix[2] $\mathbb{M}$ takes $\mathbf{x}$ to $\tilde{\mathbf{x}}$, so $\tilde{\mathbf{x}} = \mathbb{M}\mathbf{x}$. It will be useful to characterize the values that $\mathbb{M}$ could take. For instance, we can rotate $\mathbf{x} = (x_1, x_2)^\top$ to $\tilde{\mathbf{x}} = (0, \tilde{x}_2)^\top$ or $\tilde{\mathbf{x}} = (\tilde{x}_1, 0)^\top$ and anywhere in between, i.e. $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)^\top$. Simply rotating a vector should not change the length of the vector either, so we demand $\|\mathbf{x}\| = \|\tilde{\mathbf{x}}\|$ where the norm is the usual Euclidean norm given by

$$\sqrt{x_1^2 + x_2^2} = \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2}. \tag{1.1}$$

Another crucial property of these rotations is that any number of rotations applied to $\mathbf{x}$ will still create a two dimensional vector $\tilde{\mathbf{x}}$ with the same norm, $\tilde{\mathbf{x}} = \mathbb{M}_1\mathbb{M}_2 \cdots \mathbb{M}_n\mathbf{x}$. Lastly, we must have that for any $\mathbb{M}$ there exists an $\mathbb{M}^{-1}$. This condition stems from wanting to reverse the rotation back to the original starting point, i.e. $\mathbf{x} = \mathbb{M}\mathbb{M}^{-1}\mathbf{x} = \mathbb{M}^{-1}\mathbb{M}\mathbf{x} = \mathbb{I}\mathbf{x}$ where $\mathbb{I}$ is the unique 2×2 identity matrix. As we will later see, these restrictions on $\mathbb{M}$ amount to the following form

$$\mathbb{M}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \ \theta \in \mathbb{R}. \tag{1.2}$$

Using this form one can check[3] that the above requirements hold and find that $\mathbb{M}(\theta)$ preserves the norm for all $\theta \in \mathbb{R}$, is *closed*, is *associative*, and has a unique *identity* and *inverse*. Since every $\theta \in \mathbb{M}$ gives us an $\mathbb{M}(\theta)$ that satisfies these properties, we can make a set out of these matrices with the transformation being rotation. Thus we have stumbled across the definition of a group. ■

[2] *While it's tempting to draw a link between rotations and Euler's formula, rotations are an external transformation. Pictorially, they take a point on the sphere to another point. There is a connection between Euler's formula and the set of rotations, namely that the group $U(1)$, called the circle group, is isomorphic to the group of 2×2 rotations $SO(2)$. This will be made clearer later on.*

[3] *Find $\mathbb{M}^{-1}$. Find the determinant. Consider $\mathbb{M}(\theta)$ when $\theta = 0$. Lastly, check the norm of $\mathbb{M}(\theta)\mathbf{x}$ after doing the matrix multiplication.*

Definition 1.1. A **group**, G, is a set with a multiplication law ($\star$) obeying

1. *Closure:* $\forall g_1, g_2 \in G$, the product must be $g_1 \star g_2 \in G$.
2. *Associativity:* $\forall g_1, g_2, g_3 \in G$, $(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3)$
3. *Identity:* $\exists e \in G$ such that, $\forall g \in G$, $e \star g = g \star e = g$.
4. *Inverses:* $\forall g \in G, \exists g^{-1} \in G$ such that $g^{-1} \star g = g \star g^{-1} = e$.

Definition 1.2. A group, G, is **abelian** is if $g_2 \star g_1 = g_1 \star g_2, \forall g_1, g_2 \in G$.

Returning briefly to rotations, we see that the transformations are rotations, the pre-